NOTES TO THE OPTIMUM ALLOCATION IN TWO-STAGE CLUSTER SAMPLING

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(I). Signs used.

M : Number of clusters in a population.

m : Number of clusters drawn from the population.

N_j : Magnitude of the j-th cluster. $j=1, 2, 3, \cdots, M$.

n_j : Magnitude of the j-th cluster which are drawn. $j=1, 2, 3, \cdots, m$.

f : Sampling ratio of clusters and is equal to $\frac{m}{M}$.

g : Sampling ratio of the units of analysis in a cluster and is thought to be common to all of the clusters. That is equal to $\frac{n_j}{N_j}$.

a_j : Cost necessary to prepare the j-th cluster. $j=1, 2, 3, \cdots, M$.

A : Cost necessary to prepare all of the clusters. $A=\sum_{j=1}^{M} a_j$.

$\sigma_a{}^2$: Variance of $a_1, a_2, \cdots, a_M$.

$\bar{a}$: Mean of $a_1, a_2, \cdots, a_M$.

b_j : Cost necessary to inquire one unit of analysis in the j -th cluster and is thought to be common to all of the units of analysis in this cluster.

B : Cost necessary to inquire all of the units of analysis in the population.

$\sigma_{bN}{}^2$: Variance of $b_1N_1,\ b_2N_2, \cdots, b_MN_M$.

ρ : Coefficient of corelation between $\sum_{j=1}^{m} a_j$ and $\sum_{j=1}^{m} b_jn_j$

X_j : Total sum of the metrics of the units of analysis in the j-th cluster.

$\bar{X}$: Mean of $X_1, X_2, \cdots, X_M$.

$\sigma_j{}^2$: Variance of $X_1, X_2, \cdots, X_M$.

(II). Assumption to the cost function.

We treat the case where the cost function is represented by the following equation.

$$T=\sum_{j=1}^{m} a_j'+\sum_{j=1}^{m} b_j'n_j' \tag{1}$$

where the sign $'$ represents that every term is considered about sample.

By the elementary formulae we get (2), (3), (4) and (5).

$$E\left(\sum_{j=1}^{m} a_j'\right)=m\bar{a}=m\frac{A}{M}=fA \tag{2}$$

$$V\left(\sum_{j=1}^{m} a_j'\right)=\frac{M-m}{M-1}m\sigma_a{}^2 \tag{3}$$

$$E\left(\sum_{j=1}^{m} b_j'n_j'\right)=E\left\{g\left(\sum_{j=1}^{m} b_j'N_j'\right)\right\}=g\,E\left(\sum_{j=1}^{m} b_j'N_j'\right)$$

$$=g\,m\frac{1}{M}\sum_{j=1}^{m} b_jN_j=fg\,B \tag{4}$$

$$V\left(\sum_{j=1}^{m} b_j' n_j'\right) = V\left\{g\left(\sum_{j=1}^{m} b_j' N_j'\right)\right\} = g^2 V\left(\sum_{j=1}^{m} b_j' N_j'\right)$$

$$= g^2 \frac{M-m}{M-1} m\, \sigma_{bN}^2 \qquad (5).$$

Then

$$E(T) = E\left(\sum_{j=1}^{m} a_j'\right) + g E\left(\sum_{j=1}^{m} b_j' N_j'\right) = fA + fgB = f(A+gB) \qquad (6).$$

and

$$V(T) = V\left(\sum_{j=1}^{m} a_j'\right) + V\left(\sum_{j=1}^{m} b_j' n_j'\right) + 2\rho\sqrt{V\left(\sum_{j=1}^{m} a_j'\right)}\sqrt{V\left(\sum_{j=1}^{m} b_j' n_j'\right)}$$

$$= \frac{M-m}{M-1} m\sigma_a^2 + \frac{M-m}{M-1} m\sigma_{bN}^2 g^2 + 2\rho\sqrt{\frac{M-m}{M-1} m\sigma_a^2}\sqrt{\frac{M-m}{M-1} m\sigma_{bN}^2 g^2}$$

$$= \frac{M-m}{M-1} m\sigma_a^2 + \frac{M-m}{M-1} m\sigma_{bN}^2 g^2 + 2\rho\frac{M-m}{M-1} m\,\sigma_a\,\sigma_{bN}\, g$$

$$= \frac{1-\frac{m}{M}}{1-\frac{1}{M}} m\sigma_a^2 + \frac{1-\frac{m}{M}}{1-\frac{1}{M}} m\sigma_{bN}^2 g^2 + 2\rho\frac{1-\frac{m}{M}}{1-\frac{1}{M}} m\,\sigma_a\,\sigma_{bN}\, g \qquad (7).$$

Assuming that $1/M$ is small enough compared with 1 we get (8) from (7).

$$V(T) = (1-f) f M(\sigma_a^2 + 2\rho\, g\, \sigma_a\, \sigma_{bN} + \sigma_{bN}^2 g^2) \qquad (8).$$

From (6) and (8) we get (9).

$$\frac{V(T)}{\{E(T)\}^2} = \frac{(1-f) f(\sigma_a^2 + 2\rho\,\sigma_a\,\sigma_{bN}\, g + \sigma_{bN}^2 g^2)}{f^2(A+gB)^2}$$

$$= \left(\frac{1-f}{f} M\right)\left(\frac{\sigma_a^2 + 2\rho\,\sigma_a\,\sigma_{bN}\, g + \sigma_{bN}^2 g^2}{A^2 + 2ABg + B^2 g^2}\right) \qquad (9).$$

As the value of the fraction in the second bracket of (9) is very small compared with 1 because $A \gg \sigma_a$ and $B \gg \sigma_{bN}$, so we can consider the case where $\frac{V(T)}{\{E(T)\}^2}$ is very small compared with 1 even though $\frac{1-f}{f} M$ is multiplied and this is not an unreasonable idea.

Then we get $V(T) \ll E(T)^2$ and this leads us to think that we can use $E(T)$, instead of T, with comparatively small error.

We can rewrite the cost function T as

$$T = fA + fgB = f(A+gB) \qquad (10)$$

From now the problem of optimum allocation where the cost function is represented by (10).

As well known the variance of the total sum of the metrics of the units of analysis is

$$V(X') = m\left(\frac{1-f}{f^2}\sigma_e^2 + \frac{1-g}{f^2 g}\hat{\sigma}_w^2\right) = M\left(\frac{1-f}{f}\sigma_e^2 + \frac{1-g}{fg}\hat{\sigma}_w^2\right) \qquad (11),$$

where $v=\dfrac{1-f}{f}\sigma_e^2+\dfrac{1-g}{fg}\hat{\sigma}_w^2$ (11)′,

$$\sigma_e^2=\frac{1}{M}\sum_{j=1}^{M}(X_j-\bar{X})^2 \text{ and } \hat{\sigma}_w^2=\frac{1}{M}\sum_{j=1}^{M}\frac{N_j^2}{N_j-1}\sigma_j^2 .$$

(III. A.) *To determine the values of f and g which minimize v keeping T constant.*

$$v=\frac{1-f}{f}\sigma_e^2+\frac{1-g}{fg}\hat{\sigma}_w^2+\lambda(T-fA-fgB) \quad (12).$$

$$\frac{\partial v}{\partial f}=-\frac{\sigma_e^2 g+\hat{\sigma}_w^2(1-g)}{f^2 g}-\lambda(A+gB) \quad (13).$$

$$\frac{\partial v}{\partial g}=-\frac{\hat{\sigma}_w^2}{fg^2}-\lambda fB \quad (14).$$

$\lambda=-\dfrac{\hat{\sigma}_w^2}{f^2g^2B}$ is obtained from $\dfrac{\partial v}{\partial f}=0$, and substituting it to $\dfrac{\partial v}{\partial g}=0$ we get (15).

$$-\frac{g\sigma_e^2+(1-g)\hat{\sigma}_w^2}{f^2 g}+\frac{\hat{\sigma}_w^2}{f^2g^2B}(A+gB)=0$$

$$-gB\{g\sigma_e^2+(1-g)\hat{\sigma}_w^2\}+\hat{\sigma}_w^2(A+gB)=0$$

$$B(\sigma_e^2-\hat{\sigma}_w^2)g^2=A\hat{\sigma}_w^2$$

$$g^2=\frac{A\hat{\sigma}_w^2}{B(\sigma_e^2-\hat{\sigma}_w^2)}=\frac{A}{B}\,\frac{1}{\dfrac{\sigma_e^2}{\hat{\sigma}_w^2}-1}$$

Assuming $\sigma_e^2>\hat{\sigma}_w^2$ we get (15).

$$g=\sqrt{\frac{A}{B}}\;\frac{1}{\sqrt{\dfrac{\sigma_e^2}{\hat{\sigma}_w^2}-1}} \quad (15).$$

From (15) and (10) we get (16).

$$f=\frac{T}{A+gB}=\frac{T}{A+\sqrt{AB}\;\dfrac{1}{\sqrt{\dfrac{\sigma_e^2}{\hat{\sigma}_w^2}-1}}} \quad (16).$$

(III. B). *To determine the values of f and g which minimize T keeping v constant.*

$$T=fA+fgB+\lambda\left(\frac{1-f}{f}\sigma_e^2+\frac{1-g}{fg}\hat{\sigma}_w^2\right) \quad (17).$$

$$\frac{\partial T}{\partial f}=A+gB-\lambda\frac{g\sigma_e^2+(1-g)\hat{\sigma}_w^2}{f^2 g}.$$

$\lambda=\dfrac{f^2g^2B}{\hat{\sigma}_w^2}$ is obtained from $\dfrac{\partial T}{\partial g}=0$ and substituting it to $\dfrac{\partial T}{\partial f}=0$ we get (18).

$$A+gB-\frac{f^2g^2B}{\hat{\sigma}_w^2}\;\frac{g\sigma_e^2+(1-g)\hat{\sigma}_w^2}{f^2 g}=0.$$

$$g^2=\frac{A}{B}\ \frac{1}{\frac{\sigma e^2}{\hat{\sigma} w^2}-1}.$$

Making the same assumption like that in (1), (18) is obtained.

$$g=\sqrt{\frac{A}{B}}\ \frac{1}{\sqrt{\frac{\sigma e^2}{\hat{\sigma} w^2}-1}} \tag{18}$$

Substituting (18) to (11), (19) is obtained as the formula to find the value of f.

$$\frac{V}{M}=\frac{1-f}{f}\sigma e^2+\frac{1-g}{fg}\ \hat{\sigma} w^2$$

$$\left(\frac{V}{M}+\sigma e^2\right)fg=g\sigma e^2+(1-g)\hat{\sigma} w^2$$

$$\therefore\ f=\frac{g\sigma e^2+(1-g)\hat{\sigma} w^2}{\left(\frac{V}{M}+\sigma e^2\right)g}=\frac{\sigma e^2-\hat{\sigma} w^2}{\frac{V}{M}+\sigma e^2}+\frac{\hat{\sigma} w^2}{\left(\frac{V}{M}+\sigma e^2\right)g}$$

$$=\frac{\sigma e^2-\hat{\sigma} w^2}{\frac{V}{M}+\sigma e^2}+\frac{\hat{\sigma} w^2}{\frac{V}{M}+\sigma b^2}\sqrt{\frac{B}{A}}\ \sqrt{\frac{\sigma e^2}{\hat{\sigma} w^2}-1} \tag{19}$$

(IV). Here we treat a special case of the cost function T where $a_j=k,\ j=1,2,\cdots,M$, and T is given by the following equation.

$$T=k\sqrt{m}+(b_1'n_1'+\cdots+b_m'n_m')=k\sqrt{M}f^{\frac{1}{2}}+\sum_{j=1}^{m}b_j'n_j' \tag{1}$$

$$E(T)=k\sqrt{M}f^{\frac{1}{2}}+fgB=\sqrt{kA}\ f^{\frac{1}{2}}+fg\,B. \tag{2}$$

and $$V(T)=\frac{M-m}{M-1}m\sigma_{bN}{}^2g^2=(1-f)fg^2M\sigma_{bN}{}^2 \tag{3}$$

Assuming that $1/M$ is small enough compared with 1,

$$\frac{V\ (T)}{\{E(T)\}^2}=\frac{(1-f)fg^2M\sigma_{bN}{}^2}{(\sqrt{kA}\ f^{\frac{1}{2}}+fgB)^2}=\frac{(1-f)fg^2M\sigma_{bN}{}^2}{(\sqrt{kA}+f^{\frac{1}{2}}\ gB)^2f}$$

$$=(1-f)M\frac{g^2\sigma_{bN}{}^2}{fg^2B^2+2\sqrt{kA}\ Bf^{\frac{1}{2}}g+kA}=(1-f)M\ \frac{\sigma_{bN}{}^2}{fB^2+2\sqrt{kA}\ B\frac{f^{\frac{1}{2}}}{g}+\frac{kA}{g^2}} \tag{4}$$

By the reason like in (II) we can use $E(T)$, instead of T, with comparatively small error.

Now the problems of optimum allocation are treated.

(V.A). *To find the values of f and g which minimize V keeping T constant.*

$$v=\frac{1-f}{f}\sigma e^2+\frac{1-g}{fg}\hat{\sigma} w^2+\lambda(\sqrt{kA}\ f^{\frac{1}{2}}+fg\,B) \tag{5}$$

$$\frac{\partial v}{\partial f}=\frac{-1}{f^2}\left(\sigma_e^2+\frac{1-g}{g}\hat{\sigma}_w^2\right)+\lambda\left(\sqrt{kA}\,\frac{1}{2}f^{\frac{1}{2}}+g\,B\right).$$

$$\frac{\partial v}{\partial g}=\frac{1}{f}\cdot\frac{-1}{g^2}\cdot\hat{\sigma}_w^2+\lambda f B.$$

$\lambda=\dfrac{\sigma_w^2}{f^2g^2B}$ is obtained from $\dfrac{\partial v}{\partial g}=0$ and substituting it to $\dfrac{\partial v}{\partial f}=0$ we get (6).

$$\frac{-1}{f^2}\left(\sigma_e^2+\frac{1-g}{g}\hat{\sigma}_w^2\right)+\frac{\hat{\sigma}_w^2}{f^2g^2B}\left(\frac{1}{2}\sqrt{kA}\,f^{-\frac{1}{2}}+gB\right)=0$$

$$B(\sigma_e^2-\hat{\sigma}_w^2)g^2=\frac{1}{2}\sqrt{kA}\,\hat{\sigma}_w^2f^{-\frac{1}{2}}$$

$$f^{\frac{1}{2}}=\frac{\sqrt{kA}\,\hat{\sigma}_w^2}{2B(\sigma_e^2-\hat{\sigma}_w^2)g^2} \qquad (6).$$

Substituting (6) to $T=\sqrt{kA}\,f^{\frac{1}{2}}+fg\,B$ We get (7).

$$T=\sqrt{kA}\,\frac{\sqrt{kA}\,\hat{\sigma}_w^2}{2B(\sigma_e^2-\hat{\sigma}_w^2)g^2}+g\,B\frac{kA\hat{\sigma}_w^4}{4B^2(\sigma_e^2-\hat{\sigma}_w^2)^2g^4}.$$

$$4TB\,(\sigma_e^2-\hat{\sigma}_w^2)g^3-2kA\,\hat{\sigma}_w^2g-\frac{kA\hat{\sigma}_w^4}{\sigma_e^2-\sigma_w^2}=0$$

$$4TB\left(\frac{\sigma_e^2}{\hat{\sigma}_w^2}-1\right)g^3-2kAg-\frac{kA}{\dfrac{\sigma_e^2}{\hat{\sigma}_w^2}-1}=0 \qquad (7).$$

Putting $\dfrac{\sigma_e^2}{\hat{\sigma}_w^2}-1=h$, the above equation is transformed as follows.

$$4TBhg^3-2kAg-\frac{kA}{h}=0$$

$$g^3-\frac{kA}{2TBh}g-\frac{kA}{4TBh^2}=0 \qquad (8).$$

By the method used in the last essay the equation (8) is solved approximately without explanations.

$$g^3-\frac{kA}{2TBh}g-z=0$$

$$\left(3g^2-\frac{kA}{2TBh}\right)dg=dz$$

$$dg=\frac{dz}{3g^2-\dfrac{kA}{2TBh}}.$$

$$g=\sqrt{\frac{kA}{2TBh}}+\frac{dz}{3\dfrac{kA}{2TBh}-\dfrac{kA}{2TBh}}$$

$$=\sqrt{\frac{kA}{2TBh}}+\frac{TBh}{kA}\,\frac{kA}{4TBh^2}$$

$$=\sqrt{\frac{kA}{2TBh}}+\frac{1}{4h} \qquad (9)$$

This is an approximation of the root. Substituting (9) to (6) we get (10).

$$f=\frac{kA\hat{\sigma}_w^4}{4B^2(\sigma_e^2-\hat{\sigma}_w^2)}\frac{1}{g^2}=\frac{kA\hat{\sigma}_w^2}{4B^2h}\frac{1}{\sqrt{\frac{kA}{2TBh}}+\frac{1}{4h}} \qquad (10).$$

(V.B). *To find the values of f and g which minimize T keeping V constant.*

$$T=\sqrt{kA}\,f^{\frac{1}{2}}+fgB+\lambda\left(\frac{1-f}{f}\sigma_e^2+\frac{1-g}{fg}\hat{\sigma}_w^2\right) \qquad (11).$$

$$\frac{\partial T}{\partial f}=\frac{1}{2}\sqrt{kA}\,f^{-\frac{1}{2}}+gB+\lambda\left(\frac{-1}{f^2}\sigma_e^2+\frac{1-g}{g}\frac{-1}{f^2}\hat{\sigma}_w^2\right).$$

$$\frac{\partial T}{\partial g}=fB+\lambda\frac{1}{f}\frac{-1}{g^2}\hat{\sigma}_w^2.$$

$\lambda=\frac{f^2g^2B}{\hat{\sigma}_w^2}$ is obtained from $\frac{\partial T}{\partial g}=0$ and substituting it to $\frac{\partial T}{\partial f}=0$ we get (12).

$$\frac{1}{2}\sqrt{kA}\,f^{-\frac{1}{2}}+gB-\frac{1}{f^2}\frac{f^2g^2B}{\hat{\sigma}_w^2}\left(\sigma_e^2+\frac{1-g}{g}\hat{\sigma}_w^2\right)=0$$

$$f^{-\frac{1}{2}}=\frac{2Bhg^2}{\sqrt{kA}}$$

$$\therefore f=\frac{kA}{4B^2h^2g^4} \qquad (12).$$

Substituting (12) to (11) in (II),

$$\frac{V}{M}=\left(\frac{4B^2h^2g^4}{kA}-1\right)\sigma_e^2+\frac{1-g}{g}\frac{4B^2h^2g^4}{kA}\hat{\sigma}_w^2.$$

$$VkA=4MB^2h^2\sigma_e^2g^4-MkA\sigma_e^2+4MB^2h^2\hat{\sigma}_w^2g^3-4MB^2h^2\hat{\sigma}_w^2g^4.$$

$$g^4+\frac{4MB^2h^2\hat{\sigma}_w^2}{4MB^2h^2(\sigma_e^2-\hat{\sigma}_w^2)}g^3-\frac{MkA\sigma_e^2+VkA}{4MB^2h^2(\sigma_e^2-\hat{\sigma}_w^2)}=0$$

$$g^4+\frac{1}{h}g^3-\frac{kA(M\sigma_e^2+V)}{4MB^2h^3\hat{\sigma}_w^2}=0 \qquad (13).$$

Neglecting the first term we get (14).

$$g=\left\{\frac{kA(M\sigma_e^2+V)}{4MB^2h^2\hat{\sigma}_w^2}\right\}^{\frac{1}{3}} \qquad (14).$$

Substituting (14) to (12) we get (15).

$$f=\frac{kA}{4B^2h^2}\left\{\frac{4MB^2h^2\hat{\sigma}_w^2}{kA(M\sigma_e^2+V)}\right\}^{\frac{4}{3}} \qquad (15).$$

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